COMPLEX NUMBERS

Introduction

$x^2 + 1 = 0$ is a quadratic equation, whose discriminant is < 0, so it has no real roots. Further $x^2 = -1$ or $x = \pm\sqrt{-1}$ where, $\sqrt{-1}$ and $-\sqrt{-1}$ are not real numbers. Such numbers, which are not real, are called imaginary numbers. In other words, a number whose square is negative is known as an imaginary number e.g. $\sqrt{-1}$, $\sqrt{-2}$, etc.

Symbol i

We denote $\sqrt{-1}$ by the Greek letter i, called iota. Thus $\sqrt{-9}$ = 3i, $\sqrt{-5}$ = $i\sqrt{5}$ etc.

Powers of i

$i^0 = 1$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$. If n is a positive integer such that n > 4 on dividing n by 4, let m be the quotient and r be the remainder. Then n = 4m + r, where r = 0, 1, 2, 3 therefore,

$i^n = i^{(4m+r)} = i^{4m} . i^r = (i^4)^m . i^r = i^r$ (because $i^4 = 1$) $\Rightarrow$ $i^{4n+2} = -1$, $i^{4n+3} = -i$.

Negative integral powers of i

$$i^{-1} = \frac{1}{i} = \frac{i^3}{i^4} = i^3 = -i,\quad i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$$

$$i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = i,\quad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

For n > 4, $i^{-n} = \frac{1}{i^n} = \frac{1}{i^r}$, where r is the remainder when n is divided by 4.

Result: $i + i^2 + i^3 + i^4 = 0$, also $i^m + i^n + i^p + i^q = 0$, if m, n, p, q are consecutive integers.

Note

(i) For any two real numbers a and b, $\sqrt{a}\times\sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is non-negative.

(ii) If a and b are positive real numbers, then $\sqrt{-a}\times\sqrt{-b} = -\sqrt{ab}$

Complex Number

The number of the form a + ib, where a and b are real numbers and $i = \sqrt{-1}$, is known as complex number.
We denote a complex number by z = a + ib, where a is called **real part** and b is called **imaginary part** of z, and we write : Re (z) = a and Im(z) = b
Set of complex numbers (C) = { z : z = a + ib, where a, b $\in$ R}.
For example : (4 + 7i), $(-2 + i\sqrt{3})$, $(\sqrt{3} - i / 7)$, (2 + 0i) are complex numbers.
Every real number is a complex number. If u$\in$R, we can write, u = u + 0i. Therefore u is a complex number with a real part u and imaginary part is 0. Therefore real numbers are subset of complex numbers.

Purely real and Purely imaginary complex numbers

A complex number z is purely real if Im (z) = 0 and is purely imaginary if Re(z) = 0

Complex numbers as ordered pairs

Corresponding to each complex number $z = (x + i\,y)$, there is associated a unique ordered pair (x, y) of real numbers. So, we may denote a complex number $(x + i\,y)$ by (x, y)

e.g. $2 + 3i = (2, 3)$ and $i = (0, 1)$ etc.

Thus $C = \{(x, y) : x, y \in R\}$

Equality of two Complex Numbers

Two complex numbers are equal only when their real and imaginary parts are separately equal.

Thus $a + i\,b = c + i\,d \Leftrightarrow a = c$ and $b = d$

Theorem 1 : A real number can never be equal to an imaginary number.

Proof. Let $a, b \in R$ and let $a = b\,i$, where $b \neq 0$

Now, $a = b\,i \Leftrightarrow a^2 = -b^2 \Leftrightarrow a^2 + b^2 = 0 \Leftrightarrow a = 0$ and $b = 0$

This contradicts the hypothesis, $b \neq 0$

Hence, $a \neq b\,i$.

Theorem 2 : $a + i\,b = 0 \Leftrightarrow a = 0$ and $b = 0$.

Proof. $a + i\,b = 0 \Leftrightarrow a = -i\,b \Leftrightarrow a^2 = -b^2 \Leftrightarrow a^2 + b^2 = 0 \Leftrightarrow a = 0$ & $b = 0$.

Theorem 3 : Complex numbers does not hold the *property of order*. i.e., $a_1 + ib_1$, < or > $a_2 + ib_2$ is meaningless.

Proof. Let us compare i and 0.

If $i = 0$, then $i^2 = 0^2$ or $-1 = 0$, which is false

If $i > 0$, then $i^2 > i.0$ or $-1 > 0$, which is false

If $i < 0$, then $i^2 > i.0$ or $-1 > 0$, which is false

Hence, we can not compare any two complex numbers.

Also we can say i is neither positive nor negative .

Addition of Complex Numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then their sum $z_1 + z_2$ is defined as the complex number $(a_1 + a_2) + i(b_1 + b_2)$.

$\therefore$ $\text{Re}\,(z_1 + z_2) = \text{Re}(z_1) + \text{Re}\,(z_2)$

$\text{Im}\,(z_1 + z_2) = \text{Im}\,(z_1) + \text{Im}\,(z_2)$

Properties of Addition on C

Result 1 : If $z_1, z_2 \in C$, then $z_1 + z_2 \in C$ **(Closure property)**

Result 2 : If $z_1, z_2 \in C$, then $z_1 + z_2 = z_2 + z_1$ **(Commutative law)**

Result 3 : If $z_1, z_2, z_3 \in C$, then $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ **(Associative law).**

Result 4 : Existence of additive identity of $z \in C$:

For each $z \in C$ there exists $0 + i0 \in C$, such that $z + 0 + 0i = z$

Result 5 : Existence of additive inverse of $z \in C$:

For each $z \in C$ there exists $-z \in C$ such that $z + (-z) = (-z) + z = 0$

Result 6 : Cancellation law for addition in C : If $z_1, z_2, z_3 \in C$, then

$z_1 + z_3 = z_2 + z_3 \Rightarrow z_1 = z_2$ and $z_1 + z_2 = z_1 + z_3 \Rightarrow z_2 = z_3$.

Subtraction of Complex Numbers

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then $z_1 - z_2 = z_1 + (-z_2)$

$= (a_1 + ib_1) + (-a_2 - ib_2)$
$= (a_1 - a_2) + i(b_1 - b_2)$

Multiplication of Complex Numbers

Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, then
$z_1z_2 = (a_1 + ib_1)(a_2 + ib_2)$
$= (a_1a_2 - b_1b_2) + i\,(a_1b_2 + a_2b_1)$
i.e., $z_1z_2 = [\text{Re}(z_1)\,\text{Re}(z_2) - \text{Im}(z_1)\,\text{Im}(z_2)] + i\,[\text{Re}(z_1)\,\text{Im}(z_2) + \text{Re}(z_2)\,\text{Im}(z_1)]$

Properties of Multiplication on C

Result 1 : If $z_1, z_2 \in C$, then $z_1 . z_2 \in C$ **(Closure property)**
Result 2 : If $z_1, z_2 \in C$, then $z_1z_2 = z_2z_1$ **(Commutative law)**
Result 3 : If $z_1, z_2, z_3 \in C$, then $z_1(z_2 .z_3) = (z_1.z_2)\, z_3$ **(Associative law)**
Result 4 : **Existence of multiplicative identity in C :**
If $z \in C$, then $1 \in C$ is multiplicative identity because $z.\,1 = 1.\,z = z$.
Result 5 : **Existence of multiplicative inverse in C**.
If $z \in C$, $z \neq 0$ then $1/z$ or z^{-1} is multiplicative inverse because $z.1/z = 1/z.\,z = 1$.
Result 6 : **Distributive law** :
If $z_1, z_2, z_3 \in C$, then $z_1 . (z_2 + z_3) = z_1\, z_2 + z_1 . z_3$.

Multiplicative Inverse (Reciprocal) of a Complex Number

If $z \neq 0$ is any complex number, then its multiplicative inverse is denoted by $\frac{1}{z}$ or z^{-1}.

If $z = a + ib$, then $\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2}$

$$\Rightarrow \quad z^{-1} = \frac{a}{a^2+b^2} + i\left(\frac{-b}{a^2+b^2}\right)$$

Division of Complex Numbers

The division of a complex number z_1 by a non zero complex number z_2 is defined as the multiplication of z_1 by the multiplicative inverse of z_2 and is denoted by $\frac{z_1}{z_2}$.

If $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$,

$$\text{Then } \frac{z_1}{z_2} = \frac{a_1+ib_1}{a_2+ib_2} = (a_1 + ib_1)\left(\frac{1}{a_2+ib_2}\right) = (a_1 + ib_1)\left[\frac{a_2}{a_2^2+b_2^2} + \frac{i(-b_2)}{a_2^2+b_2^2}\right]$$

$$= \left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right) + i\left(\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right)$$

Conjugate of Complex Number

a + bi and a – bi (a, b are real numbers) are said to be the **complex conjugate** of each other.

(Here the complex conjugate is obtained by just changing the sign of i).
If a + bi is denoted by z, then the complex conjugate of z is denoted by $\bar{z}$ and is given by $\bar{z}$ **= a – bi.**

Properties of Conjugate

If z, z_1, z_2 are complex numbers, then

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\,\text{Re}\,(z)$

(iii) $z - \bar{z} = 2i\,\text{Im}\,(z)$

(iv) $z = \bar{z} \Leftrightarrow$ z is purely real

(v) $z + \bar{z} = 0 \Rightarrow$ z is purely imaginary.

(vi) $z\bar{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$

(vii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(viii) $\overline{z_1 - z_2} = \overline{z_1}\,\overline{z_2}$

(ix) $\overline{z_1 z_2} = \overline{z_1} + \overline{z_2}$

(x) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$

Modulus of A Complex Number

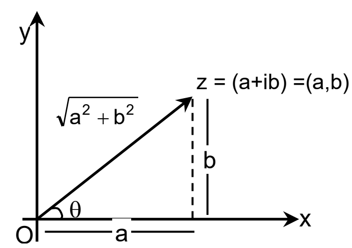
Modulus of a complex number z = a + ib, denoted by | z | is defined as

$|z| = \sqrt{(a^2 + b^2)}$ i.e., $|z| = \sqrt{(\text{Re}(z))^2 + (\text{Im}(z))^2}$

Clearly, $|z| \geq 0 \ \forall\ Z \in C$.

Example : $|5| = |5 + 0i| = \sqrt{(5^2 + 0^2)} = 5$;

$|2 + 3i| = \sqrt{(4+9)} = \sqrt{13}$



Properties of Modulus

For all complex numbers z, z_1, z_2, we have

(1) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\text{Re}(z) = \text{I}_m(z) = 0$

(2) $|z| \geq 0$

(3) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(4) $z.\bar{z} = |z|^2$

(5) $-|z| \leq \text{Re}\,(z) \leq |z|$

(6) $-|z| \leq \text{Im}\,(z) \leq |z|$.

(7) $|z_1 z_2| = |z_1|.|z_2|$

(8) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$; $z_2 \neq 0$

(9) $z_1\overline{z_2} + \overline{z_1}z_2 = 2\text{Re}(z_1.\overline{z_2}) = 2\text{Re}(\overline{z_1}z_2)$

(10) $|z^2| = |z|^2$

(11) $|z_1.z_2..........z_n| = |z_1|\,|z_2|..........|z_n|$

(12) $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle Inequality)

(13) $|z_1 - z_2| \geq ||z_1| - |z_2||$

(14) $|z_1 + z_2 + ...+ z_n| \leq |z_1| + |z_2| + ...|z_n|$.

(15) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

(16) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1.\bar{z}_2)$; $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1.\,\bar{z}_2)$

(17) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(18) $|a z_1 - b z_2|^2 + |b z_1 + a z_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$, where a, b $\in$R.

Illustration 1

Prove that :

(i) $i^{653} = i$ **(ii)** $\left(i^{47} + \frac{1}{i^{125}}\right) = -2i$ **(iii)** $i^{104} + i^{109} + i^{114} + i^{119} = 0$

Clearly, $i^n = 1$ when n is a multiple of 4.

(i) $i^{(653)} = i^{(4 \times 163 + 1)} = i^{(4 \times 163)} \times i^{(1)} = i$

(ii) $\left(i^{47} + \frac{1}{i^{125}}\right) = \left(i^{4\times11+3} + \frac{1}{i^{4\times31+1}}\right) = \left(i^3 + \frac{1}{i}\right) = \left(i^3 + \frac{i^3}{i^4}\right) = 2i^3 = -2i$

(iii) $i^{104} + i^{109} + i^{114} + i^{119} = i^{104} \times (1 + i^5 + i^{10} + i^{15})$

$= i^{104} \times (1 + i^{4+1} + i^{4\times2+2} + i^{4\times3+3}) = i^{104} \times (1 + i + i^2 + i^3) = i^{104} \times (1 + i - 1 - i) = 0.$

Illustration 2

If n is a positive integer, prove that :

(i) $(-\sqrt{-1})^{4n+3} = i$ **(ii)** $\left(\frac{1+i}{1-i}\right)^{4n+1} = i$

(i) $(-\sqrt{-1})^{4n+3} = (-i)^{4n+3} = (-i)^{4n} \cdot (-i)^3 = -i^3 = i$

(ii) $\left(\frac{1+i}{1-i}\right)^{4n+1} = \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^{4n+1} = \left[\frac{(1+i)^2}{2}\right]^{4n+1} = \left(\frac{1+i^2+2i}{2}\right)^{4n+1} = i^{4n+1} = i^{4n} \cdot i^1 = i.$

Illustration 3

For what values of x and y are the complex numbers $z_1 = x^2 - 7x + 9yi$ and $z_2 = y^2i + 20i - 12$ equal?

Since $z_1 = z_2$

$\Rightarrow$ $x^2 - 7x + 9yi = y^2i + 20i - 12$

$\Rightarrow$ $x^2 - 7x = -12$ and $9y = y^2 + 20$

$\Rightarrow$ $x^2 - 7x + 12 = 0$ and $y^2 - 9y + 20 = 0$

$\Rightarrow$ $(x - 3)(x - 4) = 0$ and $(y - 4)(y - 5) = 0$

$\Rightarrow$ $x = 3, x = 4$ and $y = 4, y = 5$

Hence solution set for x and y is given by:

$S = \{(3, 4), (3, 5), (4, 4), (4, 5)\}$

Illustration 4

For any $z_1, z_2 \in C$, prove that $|z_1 . z_2| = |z_1| |z_2|$

$|z_1 . z_2|^2 = (z_1.z_2)\,(\overline{z_1.z_2})$

$= (z_1 . z_2)(\overline{z_1}.\overline{z_2}) = (z_1.\overline{z_1})(z_2.\overline{z_2}) = |z_1|^2 |z_2|^2$

$= (|z_1||z_2|)^2$ and $0 \le |z_1 . z_2|$ and $0 \le |z_1| |z_2|$

$\therefore$ $|z_1 z_2| = |z_1| |z_2|$

$\therefore$

Illustration 5

For any $z \in C$, show that $|z^2| = |z|^2$ and $|z|^2 \neq z^2$

Let $z = x + iy$, then $z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$

$\therefore$ $|z^2| = |(x^2 - y^2) + i(2xy)|$

$= \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{(x^2 + y^2)^2} = x^2 + y^2 = |z|^2$

$z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi$

$\therefore$ $|z|^2 \neq z^2$

Note : $|z|^2$ is real and z^2 is complex.

Illustration 6

For any z_1, $z_2 \in C$ show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = (z_1 + z_2).\overline{(z_1 + z_2)} + (z_1 - z_2).\overline{(z_1 - z_2)}$

$= (z_1 + z_2).(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2).(\bar{z}_1 - \bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2$

$= 2z_1.\bar{z}_1 + 2z_2.\bar{z}_2 = 2|z_1|^2 + 2|z_2|^2$.

Illustration 7

Compute : $\frac{2-i}{3+2i} + \frac{1+i}{4-i}$

The given expression is $\frac{2-i}{3+2i} + \frac{1+i}{4-i} = \frac{(2-i)}{(3+2i)}\frac{(3-2i)}{(3-2i)} + \frac{(1+i)}{(4-i)}\frac{(4+i)}{(4+i)}$

$= \frac{6+2i^2-7i}{13} + \frac{5i+4+i^2}{17} = \frac{4-7i}{13} + \frac{3+5i}{17}$, $(\because i^2 = -1)$

$= \frac{68-119i+39+65i}{13 \times 17} = \frac{107}{221} - \frac{54\,i}{221}$

Illustration 8

Compute : $(1 - i)^{-1}$

We have, $(1 - i)^{-1} = \frac{1}{(1-i)} = \frac{1}{(1-i)} \times \frac{1+i}{1+i} = \frac{1+i}{2} = \left(\frac{1}{2} + \frac{1}{2}i\right)$

Illustration 9

$(x + iy)^{1/3} = a + ib$, x, y, a , b $\in$ R. **Show that $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$**

We have $(x + iy)^{1/3} = a + ib$.
Cubing both sides, we get
$x + iy = a^3 + 3a^2ib + 3ai^2b^2 + i^3b^3 = (a^3 - 3ab^2) + i(3a^2b - b^3)$.
Equating the real and imaginary parts on both sides, we get
$x = a^3 - 3ab^2$, $y = 3a^2b - b^3$

$\therefore \quad \frac{x}{a} + \frac{y}{b} = \frac{a(a^2 - 3b^2)}{a} + \frac{b(3a^2 - b^2)}{b}$
$= a^2 - 3b^2 + 3a^2 - b^2 = 4(a^2 - b^2)$

Illustration 10

Express each one of the following in the form (A + i B) :

(i) $\left(\frac{1+2i}{2+i}\right)^2$ **(ii)** $\left(\frac{1}{1-2i} + \frac{3}{1+4i}\right)\left(\frac{3+4i}{2-4i}\right)$

(i) Given expression is $\left(\frac{1+2i}{2+i}\right)^2 = \frac{(1+2\,i)^2}{(2+i)^2} = \frac{(-3+4\,i)}{(3+4\,i)} = \frac{(-3+4\,i)}{(3+4\,i)} \times \frac{(3-4\,i)}{(3-4\,i)} = \frac{(7+24\,i)}{25} = \left(\frac{7}{25}\right) + i\left(\frac{24}{25}\right)$

(ii) Given expression is

$= \frac{(1+4\,i)+(3-6\,i)}{(1-2\,i)(1+4\,i)} \cdot \left(\frac{3+4\,i}{2-4\,i}\right) = \frac{(4-2\,i)(3+4\,i)}{(9+2\,i)(2-4\,i)}$

$= \frac{(10+5\,i)}{(13-16\,i)} \times \frac{(13+16\,i)}{(13+16\,i)} = \frac{25(2+9\,i)}{425} = \left(\frac{2}{17}\right) + i\left(\frac{9}{17}\right)$

Illustration 11

If $(x + i\,y) = \frac{3}{(2+\cos\theta + i\sin\theta)}$, prove that : $(x - 1)(x - 3) + y^2 = 0$

$(x + i\,y) = \frac{3}{(2+\cos\theta+i\sin\theta)} \times \frac{(2+\cos\theta - i\sin\theta)}{(2+\cos\theta - i\sin\theta)} = \frac{(6+3\cos\theta - 3\,i\sin\theta)}{(5+4\cos\theta)}$

$\therefore \quad x = \left(\frac{6+3\cos\theta}{5+4\cos\theta}\right)$ and $y = \frac{-3\sin\theta}{(5+4\cos\theta)}$

So, $(x-1)(x-3)+y^2 = \left(\frac{6+3\cos\theta}{5+4\cos\theta}-1\right)\left(\frac{6+3\cos\theta}{5+4\cos\theta}-3\right)+\frac{9\sin^2\theta}{(5+4\cos\theta)^2}$

$= \frac{(1-\cos\theta)}{(5+4\cos\theta)}\cdot\frac{-9(1+\cos\theta)}{(5+4\cos\theta)}+\frac{9\sin^2\theta}{(5+4\cos\theta)^2} = 0$

Square Roots of a Complex Number

Square root of complex number a + ib is written as $\sqrt{(a+ib)}$

Let $\sqrt{(a+ib)} = x + iy$, where x and y are real numbers.

$\Rightarrow \quad (a+ib) = (x+iy)^2 \Rightarrow a + ib = x^2 - y^2 + 2xyi$

$\Rightarrow \quad x^2 - y^2 = a$ and $2xy = b$

Now $\quad (x^2+y^2) = \sqrt{[(x^2-y^2)^2+4x^2y^2]} = \sqrt{(a^2+b^2)}$

Solving $\quad x^2 - y^2 = a$ and $x^2 + y^2 = \sqrt{(a^2+b^2)}$, we get

$$x = \pm\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} \text{ and } y = \pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$$

The sign of x and y can be determined from the expression xy = b/2. This will give two square roots of a + bi.

$\therefore$ If z = a + ib, b is positive, then

$$\sqrt{z} = \pm\left\{\sqrt{\frac{1}{2}[|z|+\text{Re}(z)]}+i\sqrt{\frac{1}{2}[|z|-\text{Re}(z)]}\right\}$$

and if b is negative, then $\sqrt{z} = \pm\left\{\sqrt{\frac{1}{2}[|z|+\text{Re}(z)]}-i\sqrt{\frac{1}{2}[|z|-\text{Re}(z)]}\right\}$

Note

If $\sqrt{z} = \pm z_1$ then $\sqrt{\bar{z}} = \pm\bar{z}_1$

Illustration 12

Find the square roots of (i) –24 + 10i (ii) –i

(i) Let $\sqrt{-24+10i} = x+iy$, $\quad x, y \in R$

Squaring both sides, we get $-24 + 10i = x^2 - y^2 + 2xyi$

Equating the real and imaginary parts separately, we get

$x^2 - y^2 = -24$ and $2xy = 10$.

Eliminating y, $x^2 - \left(\frac{5}{x}\right)^2 = -24$

i.e. $\quad x^4 + 24x^2 - 25 = 0$

i.e. $\quad (x^2-1)(x^2+25) = 0$

$\therefore \quad x^2 = 1, x^2 \neq -25 \qquad (\because$ x is real)

$\therefore \quad x = \pm 1. \qquad \therefore y = \frac{5}{x} = \pm 5$

$\therefore \quad$ Square roots of $-24 + 10i$ are $\pm(1 + 5i)$

(ii) Square root of –i ; $\quad$ Let $\sqrt{-i} = x + iy \quad x, y \in R$

Squaring both sides, $-i = x^2 - y^2 + 2xyi$

Equating real and imaginary parts separately, we get

$x^2 - y^2 = 0$ and $2xy = -1$

$\therefore \quad x^2 = y^2$ and $x = -\dfrac{1}{2y}$

$\therefore \quad (-\dfrac{1}{2y})^2 = y^2 \Rightarrow y^4 = \dfrac{1}{4}$

$\therefore \quad y^2 = +\dfrac{1}{2} \quad \Rightarrow \quad y = \pm\dfrac{1}{\sqrt{2}} \qquad (\because y^2 \not< 0)$

$\therefore \quad x = -\dfrac{1}{2(\pm\frac{1}{\sqrt{2}})} = -\dfrac{1}{\pm\sqrt{2}} = \mp\dfrac{1}{\sqrt{2}}$

$\therefore \quad$ Square root of $-i = \mp\dfrac{1}{\sqrt{2}} \pm \dfrac{1}{\sqrt{2}}i = \mp\dfrac{1}{\sqrt{2}}(1-i)$.

Illustration 13

Find the square root of $-3 + 4i$.

To find the square root of $a + ib$, we have the formula

$$(a+ib)^{1/2} = \begin{cases} \pm\left[\sqrt{\dfrac{|z|+a}{2}} + i\sqrt{\dfrac{|z|-a}{2}}\right] & ,\text{if } b > 0 \\ \pm\left[\sqrt{\dfrac{|z|+a}{2}} - i\sqrt{\dfrac{|z|-a}{2}}\right] & ,\text{if } b < 0 \end{cases}$$

Let $z = -3 + 4i = a + ib$

We have $a = -3$, $b = 4 > 0$

$|z| = \sqrt{9+16} = 5$

Hence $z^{1/2} = \pm\left[\sqrt{\dfrac{5+(-3)}{2}} + i\sqrt{\dfrac{5-(-3)}{2}}\right] = \pm(1 + 2i)$

Illustration 14

Find the value of : $\sqrt{-15+8i} - \sqrt{-15-8i}$

Using the formula $(\sqrt{a+ib} - \sqrt{a-ib})^2 = (a+ib) + (a-ib) - 2\sqrt{a+ib}.\sqrt{a-ib} = 2\left[a - \sqrt{a^2+b^2}\right]$

$\therefore \quad (\sqrt{a+ib} - \sqrt{a-ib}) = \pm\sqrt{2\left(a - \sqrt{a^2+b^2}\right)}$

$\therefore \quad (\sqrt{-15+8i} - \sqrt{-15-8i}) = \pm\sqrt{2\left\{-15 - \sqrt{225+64}\right\}} \qquad$ (Taking $a = -15$, $b = -8$)

$= \pm\sqrt{-64} = \pm 8i$.

Illustration 15

Find the value of $x^3 + 7x^2 - x + 16$ when $x = 1 + 2i$

$x = 1 + 2i \Rightarrow x - 1 = 2i \Rightarrow (x - 1)^2 = 4i^2$

$\Rightarrow$ $x^2 - 2x + 1 = -4 \Rightarrow x^2 - 2x + 5 = 0$

Now $x^3 + 7x^2 - x + 16$

$= x(x^2 - 2x + 5) + 9(x^2 - 2x + 5) + (12x - 29)$

$= (x + 9)(x^2 - 2x + 5) + (12x - 29)$

$= (x + 9)(0) + 12x - 29$ $[\because x^2 - 2x + 5 = 0]$

$= 0 + 12(1 + 2i) - 29 = -17 + 24i$ $[\because x = 1 + 2i]$

Practice Assignment – I

1 Simplify the following :

(i) $[i^{327} + i^{-63} + (\sqrt{-1})^7]$

(ii) $(1+i)^4\left(1+\frac{1}{i}\right)^4$

(iii) $(i^n + i^{n+1} + i^{n+2} + i^{n+3})$

(iv) $\left(\frac{3+i}{2-i} + \frac{3-i}{2+i}\right)$

(v) $\left(\frac{3}{1+i} - \frac{2}{2-i} + \frac{2}{1-i}\right)$

(vi) $i^{49} + i^{68} + i^{89} + i^{110}$

(vii) $i^{30} + i^{80} + i^{120}$

(viii) $i + i^2 + i^3 + i^4$

(ix) $i^5 + i^{10} + i^{15}$

(x) $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$

(xi) $1 + i^2 + i^4 + i^6 + i^8 + \ldots\ldots + i^{20}$

2. Express $(5 - 3i)^3$ in the form $a + ib$.

3. On substituting $a = -1$ and $b = -1$ in the formula $\sqrt{a}\,.\,\sqrt{b} = \sqrt{ab}$, we get $1 = -1$. What is wrong with the formula? Compute: $\sqrt{-2}.\sqrt{-3}$.

4. Simplify:

(i) $(5-\sqrt{-7})(5+\sqrt{-7})(1-\sqrt{-1})(1+\sqrt{-1})$

(ii) $[(\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)]^{-2}$

(iii) $\frac{\sqrt{3+4i}+\sqrt{3-4i}}{\sqrt{3+4i}-\sqrt{3-4i}}$

(iv) $(1-i)^{-1}$

(v) $[(\sqrt{3}+\sqrt{-1})(\sqrt{3}-\sqrt{-1})]^{-3/2}$

5. Find the conjugates of the following complex numbers:

(i) $4-5i$ (ii) $\frac{1}{3+5i}$ (iii) $\frac{1}{1+i}$

(iv) $\frac{(3-i)^2}{2+i}$ (v) $\frac{(1+i)(2+i)}{3+i}$

6. If $(p+iq) = \frac{(a+i)^2}{(2a-i)}$, prove that : $(p^2+q^2) = \frac{(a^2+1)^2}{(4a^2+1)}$

7. If $(a^2+b^2) = 1$, show that : $\left(\frac{1+b+ia}{1+b-ia}\right) = (b+ia)$.

8. (i) If $z_1 = (1-i)$ and $z_2 = (-2+4i)$, show that : $\text{Im}\left(\frac{z_1z_2}{\bar{z}_1}\right) = 2$

(ii) If $z_1 = 2 - I$, $z_2 = -2 + I$, fine $\text{Re}\left(\frac{z_1z_2}{\bar{z}_1}\right)$ and $\text{Im}\left(\frac{1}{z_1\bar{z}_1}\right)$.

9. With what complex number must $(\sqrt{2}-3i)$ be multiplied to obtain $(7-5\sqrt{2}\,i)$?

10. If $(a+ib)(x+iy) = (a^2+b^2)\,i$, find x and y.

11. Prove that : $(x+1+i)(x+1-i)(x-1+i)(x-1-i) = (x^4+4)$.

12. If $z = \frac{-3+\sqrt{-5}}{2}$, find the value of $(2z^3+12z^2+25z+29)$. Show that it will be unaltered, if $z = \frac{-3-\sqrt{-5}}{2}$

13. Express each one of the following in the form $(a+ib)$:

(i) $\frac{1}{(-2+\sqrt{-3})}$ (ii) $\frac{(2+3i)^2}{(2-i)}$ (iii) $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

(iv) $\left(\frac{1-i}{1+i}\right)^{50}$ (v) $\frac{(a+ib)^2}{(a-ib)} - \frac{(a-ib)^2}{(a+ib)}$ (vi) $\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}$

(vii) $\left[\left(\frac{1}{3}+i\frac{7}{3}\right)+\left(4+i\frac{1}{3}\right)-\left(-\frac{4}{3}+i\right)\right]$ (vii) $(1-i)^4$ (ix) $\left(\frac{1}{3}+3i\right)^3$

(x) $\left(-2-\frac{1}{3}i\right)^3$

14. If $x = -5+2\sqrt{-4}$, find the value of $(x^4+9x^3+35x^2-x+4)$.

15. Solve : (i) $\bar{z} = iz^2$ (ii) $z^2+|z| = 0$

16. If $(x+iy) = \sqrt{\frac{a+ib}{c+id}}$, Prove that :

$(x^2+y^2)^2 = \left(\frac{a^2+b^2}{c^2+d^2}\right)$.

17. If $iz^3 + z^2 - z + i = 0$ show that $|z| = 1$

18. Find the real values of x and y satisfying

(i) $(x + iy)(2 - 3i) = (4 + i)$

(ii) $\dfrac{(1+i)x-2i}{(3+i)} + \dfrac{(2-3i)y+i}{3-i} = i$

(iii) $(x^4 + 2x\,i) - (3x^2 + y\,i) = (3 - 5i) + (1 + 2y\,i)$

(iv) $\sqrt{x^2 - 2x + 8} + (x + 4)i = (2 + i)y$.

19. If $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = (a + bi)$, show that :
$2.\,5.\,10.17 \ldots (1 + n^2) = (a^2 + b^2)$

20. (i) Find real values of x and y for which the complex numbers $-3 + i\,x^2y$ and $x^2 + y + 4i$ are conjugate of each other.

(ii) Find real x and y if $(x + iy)(3 + 5i)$ is conjugate of $-6 - 24i$.

21. Show that

$$\frac{\sqrt{x^2+1}+xi}{\sqrt{x^2+1}-xi}+\frac{\sqrt{x^2+1}-xi}{\sqrt{x^2+1}+xi} = \frac{2}{2x^2+1}, \text{ where } x \in R.$$

22. If $z = x + iy$; $x, y \in R$ and $w = \dfrac{1-iz}{z-i}$, show that $|w| = 1$ implies that z is purely real.

23. Find the square roots of the following numbers :

(i) $-7 + 24\sqrt{-1}$ (ii) $3 - 4\sqrt{-1}$ (iii) $-2 + 2\sqrt{3}\,i$

(iv) i (v) $-8i$ (vi) $\left(\dfrac{3+4i}{3-4i}\right)$

(vii) $\dfrac{86-27i}{2+3i}$ (viii) $-2i$ (ix) $a + i\sqrt{a^4 + a^2 + 1}$

(x) $4ab - 2(a^2 - b^2)\sqrt{-1}$

24. Evaluate :

(i) $\sqrt{16+30i} + \sqrt{16-30i}$ (ii) $\sqrt{16+30i} - \sqrt{16-30i}$

(iii) $\left(4+3\sqrt{-20}\right)^{1/2} + \left(4-3\sqrt{-20}\right)^{1/2}$

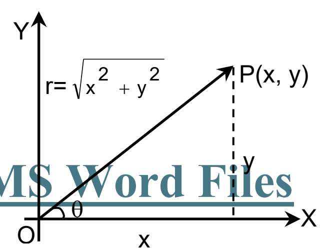
25. If z is a complex number such that $|z| = 1$, prove that $\dfrac{z-1}{z+1}$ is purely imaginary. What will be your conclusion if $z = 1$?

Representations of a Complex Number

A complex number can be represented geometrically, vectorially or trigonometrically.

1. Geometric representation : A complex number $z = x + iy$ can be represented by a point P on the complex plane (known as Argand Plane) by the ordered pair (x, y). P is called affix of $z = x + iy$.

Modulus : The absolute length OP = $\sqrt{x^2 + y^2}$ is called the modulus of the complex number x + iy which is denoted by | z |.



i.e if $z = x + iy$ then $| z | = \sqrt{x^2 + y^2}$ = the length OP.

Argument (Amplitude)

The amplitude (or argument) of the complex number is defined as the angle made by OP with the positive x-axis.

θ = amplitude z = amplitude (x + iy) = $\tan^{-1}\frac{y}{x}$.

This angle has infinitely many values differing by multiples of 2π. The unique value of θ such that $-\pi < \theta \le \pi$ is called the **principal value** of the amplitude.

Unless otherwise stated, amp z implies the principal value of the amplitude z.

Note :

(i) Amplitude of the complex number 0 is not defined.

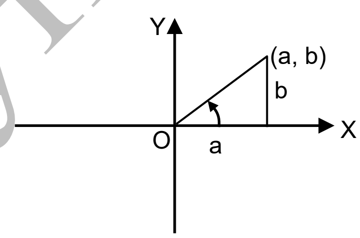
(ii) $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ and amplitude z_1 = amplitude z_2.

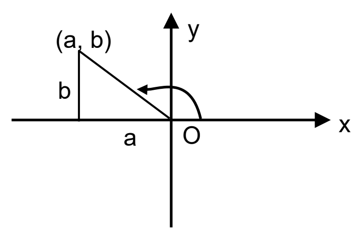
(iii) There exists a one-one correspondence between the points of the plane and the members of the set of complex numbers i.e. for every complex number z there exists uniquely a point (x, y) on the plane and for every point (x, y) of the plane there exists uniquely a complex number $z = x + iy$.

Hence $x_1 + iy_1 = x_2 + iy_2 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$.

Argument of a complex number z = a + ib for different signs of a and b

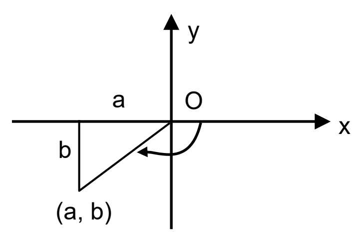
(i) (a, b) $\in$ I quadrant , $a > 0$, $b > 0$.
The principal value = amp z = $\tan^{-1}$ (b/a)
It is an acute angle and positive.



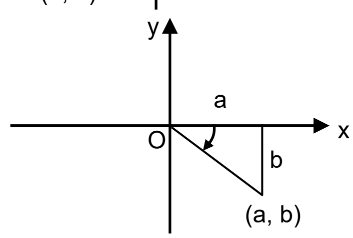


(ii) (a, b) $\in$ II quadrant, $a < 0$, $b > 0$
The principal value = amp z = $\pi - \tan^{-1}$ (b/|a|)
It is an obtuse angle and positive

(iii) (a, b) $\in$ III quadrant , $a < 0$, $b < 0$
The principal value = amp z = $-\pi + \tan^{-1}$ (b/a)
It is an obtuse angle and negative.



(iv) (a, b) $\in$ IV quadrant, $a > 0$, $b < 0$
The principal value = amp z = $-\tan^{-1}$ (|b|/a)
It is an acute angle and negative.



For instance, taking $z_1 = 3 + \sqrt{3}i$, $z_2 = -3 + \sqrt{3}i$, $z_3 = -3 - \sqrt{3}i$, $z_4 = +3 - \sqrt{3}i$, the principal values are given by

amp $z_1 = \tan^{-1}(1/\sqrt{3}) = \frac{\pi}{6}$, amp $z_2 = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{5\pi}{6}$,

amp $z_3 = -\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{5\pi}{6}$, amp$z_4 = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

2. Vectorial Representation : A complex number z can be represented by the position vector OP because a complex number depends on two things :

(i) its modulus and

(ii) its amplitude.

These are the requirements of a vector on a plane. (For a vector, amplitude is replaced by the direction of the vector i.e. inclination of the vector with the positive x-axis).

Note :

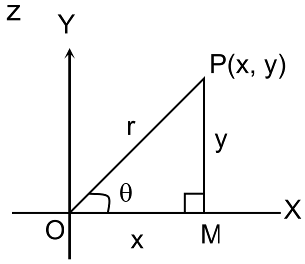
(i) There exists a one-one correspondence between the set of position vectors and the set of complex numbers.

(ii) As far as addition, subtraction and multiplication by real numbers are concerned, complex numbers are subject to the same laws as the vectors, which represent them.

(iii) If a complex number z is represented by a vector $\vec{AB}$, then | z | is the length AB and amp z is the angle which the directed line AB makes with the directed line OX.

3. Polar or Trigonometric Representation of a Complex Number

Let z = x + iy be a complex number represented by a point P(x, y) in the argand plane. Then by geometrical representation, we have OP = |z| = r and ∠POM = θ = arg z



In ΔPOM, $\cos\theta = \frac{OM}{OP} = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{PM}{OP} = \frac{y}{r} \Rightarrow y = r\sin\theta$

∴ $z = x + iy = r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta)$ or r cis θ

Here, $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$.

This form of z is called a polar form of z.

If we use general value of the argument of θ, then polar form of z is given by

$z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$ where n is an integer.

Eulerian Form of a Complex Number

We have, $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

These two are called Euler's notations. Let z be any complex number such that |z| = r and arg z = θ. Then, in polar fom, $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z = re^{i\theta}$

This form of z is known as Eulerian form.

Properties of Argument

1. $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$, where k = 0, 1 or –1

In general,

$\text{Arg}(z_1z_2z_3\ldots\ldots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \ldots\ldots + \arg(z_n) + 2k\pi$, where k = 0, 1 or –1

2. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2k\pi$, where k = 0, 1 or –1

3. $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg z + 2k\pi$, where k = 0, 1 or –1

4. $\arg(z^n) = n \arg z + 2k\pi$, where k = 0, 1 or –1

5. If $\arg\left(\frac{z_2}{z_1}\right) = \theta$, then $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$, where $k \in I$

6. $\arg(\bar{z}) = -\arg z$

7. If $\arg z = 0$ or π, then z is purely real

8. If $\arg z = \frac{\pi}{2}$ or $-\frac{\pi}{2}$, then z is purely imaginary

9. $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2} \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary

10. $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2) = \frac{-\pi}{2} \Leftrightarrow \frac{z_1}{z_2}$ is purely real

Illustration16

Find the modulus and principal arguments of the following complex numbers.

(i) $i(1+i\sqrt{3})(\sqrt{3}-i)$

(ii) $\mathbf{\frac{1+2i}{(1-i)^2-2}}$

(i) The given number in the form $a+ib$ is $-2+2\sqrt{3}\,i$

$\Rightarrow$ The magnitude is $\sqrt{(-2)^2+(2\sqrt{3})^2} = 4$ and the principal argument is $\pi - \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$

$= \pi - \tan^{-1}\sqrt{3}$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

(ii) The number in the $a+bi$ form is $-\frac{3}{4}-\frac{1}{4}i$

$\Rightarrow$ Magnitude is $\frac{\sqrt{10}}{4}$ and the argument is $-\pi+\tan^{-1}\left(\frac{1}{3}\right)$

Illustration 17

Find the modulus and principal argument of the following and express them in trigonometric and exponential forms:

(i) $1 + i$

(ii) $\dfrac{1+7i}{(2-i)^2}$

(i) Let $z = 1 + i = r(\cos\theta + i\sin\theta)$

$\therefore$ $r\cos\theta = 1$ and $r\sin\theta = 1$.

On squaring and adding, we get

$r^2 = 2 \Rightarrow r = \sqrt{2}$ $\quad\therefore\quad$ $\cos\theta = \dfrac{1}{\sqrt{2}}$ and $\sin\theta = \dfrac{1}{\sqrt{2}}$

The value of θ such that $-\pi < \theta \le \pi$, satisfying both of the above equations, is given by, $\theta = \dfrac{\pi}{4}$.

$\therefore$ $z = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

Hence, $|z| = \sqrt{2}$ and $\arg(z) = \dfrac{\pi}{4}$.

$\therefore$ Trigonometric form of z is $\sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$ and its exponential form is $\sqrt{2}e^{\frac{i\pi}{4}}$.

(ii) Let $z = \dfrac{1+7i}{(2-i)^2} = \dfrac{1+7i}{4+i^2-4i} = \dfrac{1+7i}{3-4i} = \dfrac{1+7i}{3-4i} \times \dfrac{3+4i}{3+4i} = \dfrac{-25+25i}{25} = -1+i$.

Now Let $(-1+i) = r(\cos\theta + i\sin\theta)$.

$\therefore$ $r\cos\theta = -1$ and $r\sin\theta = 1$.

On squaring and adding, we get

$r^2 = 2 \Rightarrow r = \sqrt{2}$

$\therefore$ $\cos\theta = -\dfrac{1}{\sqrt{2}}$ and $\sin\theta = \dfrac{1}{\sqrt{2}}$

The value of θ such that $-\pi < \theta \le \pi$, satisfying both of the above equations, is given by, $\theta = \dfrac{3\pi}{4}$.

$\therefore$ $z = \sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

Hence, $|z| = \sqrt{2}$ and $\arg(z) = \dfrac{3\pi}{4}$.

$\therefore$ Trigonometric form of z is $\sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$ and its exponential form is $\sqrt{2}e^{\frac{i3\pi}{4}}$.

Illustration 18

Express each of the following complex numbers in trigonometric form

(i) $(1 - \cos\theta + i\sin\theta)$

(ii) $(1 + i\tan\alpha)$, where $-\pi < \alpha < \pi$ and $\alpha \neq \pm\frac{\pi}{2}$

(iii) $\frac{(1+i)^{2n+1}}{(1-i)^{2n-1}}$, $n \in N$

(i) Let $(1 - \cos\theta + i\sin\theta) = r(\cos\phi + i\sin\phi)$. Then,
$r\cos\phi = (1 - \cos\theta)$ and $r\sin\phi = \sin\theta$
On squaring and adding, we get
$r^2 = 2(1 - \cos\theta) = 4\sin^2\frac{\theta}{2}$. So $r = 2\sin\frac{\theta}{2}$

$\therefore \cos\phi = \frac{1-\cos\theta}{2\sin(\theta/2)} = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)} = \sin\frac{\theta}{2}$

And, $\sin\phi = \frac{\sin\theta}{2\sin(\theta/2)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin(\theta/2)} = \cos\frac{\theta}{2}$

$\therefore \tan\phi = \cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$. So, $\phi = \left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

$\therefore (1 - \cos\theta + i\sin\theta) = 2\sin\frac{\theta}{2}\left[\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right]$

(ii) Let $(1 + i\tan\alpha) = r(\cos\theta + i\sin\theta)$. Then,
$r\cos\theta = 1$ and $r\sin\theta = \tan\alpha$

$\therefore r^2 = (1 + \tan^2\alpha) = \sec^2\alpha$ or $r = |\sec\alpha| = \frac{1}{|\cos\alpha|}$

Case I. When $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

In this case, $\cos\alpha$ is positive. So, $r = \frac{1}{\cos\alpha}$

$\therefore \cos\theta = \frac{1}{r} = \cos\alpha$ and $\sin\theta = \frac{\tan\alpha}{r} = \sin\alpha$

So, $(1 + i\tan\alpha) = \frac{1}{\cos\alpha}(\cos\alpha + i\sin\alpha)$

Case II. When $\left(\frac{\pi}{2} < \alpha < \pi\right)$ or $\left(-\pi < \alpha < -\frac{\pi}{2}\right)$

In this case, $\cos\alpha$ is negative, so,

$|\cos\alpha| = -\cos\alpha$. And, $r = \dfrac{-1}{\cos\alpha}$

$\therefore\ \cos\theta = \cos\alpha$ and $\sin\theta = -\sin\alpha$

i.e. $\cos\theta = \cos(\pi+\alpha)$ and $\sin\theta = \sin(\pi+\alpha)$. So, $\theta = \pi + \alpha$

$\therefore\ (1 + i\tan\alpha) = \dfrac{-1}{\cos\alpha}\,.\,[\cos(\pi+\alpha) + i\sin(\pi+\alpha)]$

(iii) $\dfrac{(1+i)^{2n+1}}{(1-i)^{2n-1}} = \dfrac{(1+i)^{2n+1}(1+i)^{2n-1}}{(1-i)^{2n-1}(1+i)^{2n-1}} = \dfrac{(1+i)^{4n}}{2^{2n-1}}$

$= \dfrac{\left[(1+i)^2\right]^{2n}}{2^{2n-1}} = \dfrac{(2i)^{2n}}{2^{2n-1}} = \dfrac{2^{2n}.(i^2)^n}{2^{2n-1}} = 2(-1)^n$

$= \begin{cases} 2, \text{ when n is even} \\ -2, \text{ when n is odd} \end{cases}$

$= \begin{cases} 2(\cos 0 + i\sin 0), \text{ when n is even} \\ 2(\cos\pi + i\sin\pi), \text{ when n is odd} \end{cases}$

Practice Assignment – II

1. Find the modulus and principal argument of the following complex numbers, and hence express each of them in the polar form:

(i) $\sqrt{3} + i$ (ii) $1 - i$

(iii) $\dfrac{1-i}{1+i}$ (iv) $\dfrac{1}{1+i}$

(v) $\dfrac{1+2i}{1-3i}$ (vi) $\dfrac{1-3i}{1+2i}$

(vii) $-1 - i\sqrt{3}$ (viii) $-\sqrt{3} + i$

(ix) -3

2. Find the principal argument of the following complex numbers:

(i) $\dfrac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}, \dfrac{\pi}{4} < \theta < \dfrac{\pi}{2}$ (ii) $\dfrac{\sqrt{3}+i}{\sqrt{3}-i}$

(iii) $\sin 120° - i\cos 120°$ (iv) $\cos 90° + \sin 90°$

(v) $\cos 70° + i\cos 20°$

3. Find the modulus of the following complex numbers

(i) $\dfrac{3+i}{(1+i)(2+i)}$ (ii) $\left(\dfrac{3}{1+i} + \dfrac{1}{1-2i}\right)\left(\dfrac{3+4i}{2-4i}\right)$

(iii) $\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}$

4. Find the modulus and principal argument of the following:

(i) $\sin 50° + i\cos 50°$ (ii) $\dfrac{1+2i}{1-(1-i)^2}$

(iii) $\tan\theta - i$ (iv) $1 - \sin\alpha + i\cos\alpha$

5. If α and β are different complex numbers with $|\beta| = 1$, then find $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$.

6. Find the number of non-zero integral solutions of the equation $|1 - i|^x = 2^x$.

7. If (a + ib) (c + id) (e + if) (g + ih) = A + iB, then show that $(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$.

8. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m.

Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$ $(a \neq 0)$ in which x^2 is the highest power of x (i.e., the highest index of x is 2) and a, b, c are any three numbers free from x is called **quadratic equation** or an equation of second degree in x. Here 'a' is called the coefficient of x^2, 'b' the coefficient of x and 'c' the constant term. If b = 0, the equation reduces to $ax^2 + c = 0$ which is called a pure quadratic.

Methods of Solving a Quadratic Equation

A quadratic equation may be solved either by factorising the left side (when the right side is zero) or by completing a square on the left side as shown below.

Factorization Method of Solving a Quadratic Equation

The principle underlying this method is that if xy = 0, then either x = 0 or y = 0

Let $ax^2 + bx + c = 0$, $a \neq 0$ be the given quadratic equation, where a, b, c are complex numbers.

If b = 0, then $ax^2 + c = 0$, i.e., $(\sqrt{a}\,x + \sqrt{-c})(\sqrt{a}\,x - \sqrt{-c}) = 0$ or $\sqrt{a}\,x + \sqrt{-c} = 0$ or $\sqrt{a}\,x - \sqrt{-c} = 0$ or $x = \pm\sqrt{-c/a}$. So, let us assume that $b \neq 0$. The method of factorization is applicable only if we can write b as the sum of two numbers whose product is ac. The value of b is changed in the given equation and the factorization is carried.

Working Rules for Solving Problems

(i) If the equation to solve is $ax^2 + c = 0$, then its roots are $\pm\sqrt{\frac{-c}{a}}$

(ii) If the equation to solve is $ax^2 + bx + c = 0$, then find two numbers ℓ and m such that $\ell + m = b$ and $\ell m = ac$. Put $b = \ell + m$ in the equation and factorize the L.H.S.

(iii) After expressing L.H.S. as the product of two linear polynomials, each linear factor equated to zero gives one root.

Illustration 19

Solve the equation : $5x^2 + 15 = 0$

We have $5x^2 + 15 = 0$

$\Rightarrow$ $5(x^2 + 3) = 0 \Rightarrow x^2 + 3 = 0$ $\Rightarrow x^2 - 3i^2 = 0$

$\Rightarrow$ $(x - \sqrt{3}i)(x + \sqrt{3}i) = 0 \Rightarrow x - \sqrt{3}i = 0$ or $x + \sqrt{3}i = 0$

$\Rightarrow$ $x = \sqrt{3}i$ or $x = -\sqrt{3}i$

$\therefore$ The roots are $\sqrt{3}i$ and $-\sqrt{3}i$

$\therefore$

Illustration 20

Solve the equation : $4x^2 - 16ix - 15 = 0$

We have $4x^2 - 16ix - 15 = 0$ ...(1)

We write $-16i = (-6i) + (-10i)$, because $(-6i)(-10i) = 60i^2 = -60 = (4)(-15)$

$\therefore$ (1) $\Rightarrow$ $4x^2 + (-6i - 10i)x - 15 = 0$

$\Rightarrow$ $4x^2 - 6ix - 10ix + 15i^2 = 0$

$\Rightarrow$ $2x(2x - 3i) - 5i(2x - 3i) = 0$

$\Rightarrow$ $(2x - 3i)(2x - 5i) = 0$

$\therefore$ $2x - 3i = 0$ or $2x - 5i = 0$

$\Rightarrow$ $x = \frac{3i}{2}$ or $x = \frac{5i}{2}$

$\therefore$ The roots are 3i/2 and 5i/2

$\therefore$

Formula Method of Solving a Quadratic Equation

The 'formula method' of solving a quadratic equation is used when the 'factorization method' is not easily applicable.

Let $ax^2 + bx + c = 0$, $a \neq 0$ ...(1)

be the given quadratic equation, where a, b, c are complex numbers.

(1) $\Rightarrow ax^2 + bx = -c \quad \Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad (\because a \neq 0)$

$$\Rightarrow \quad x^2 + 2\left(\frac{b}{2a}\right)x = -\frac{c}{a} \Rightarrow x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \Rightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

These are the required roots of the given equation.

Working Rules for Solving Problems

(i) Simplify the given equation and express it in the form $ax^2 + bx + c = 0$

(ii) Identify the values of a, b and c.

(iii) Use the formula : $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and simplify it.

(iv) The values of x are the roots of the given equation.

Illustration 21

Find the roots of the equation $2x^2 - 10x + 13 = 0$

$2x^2 - 10x + 13 = 0$

Here a = 2, b = –10, c = 13

Applying the formula :

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we get

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4.2.13}}{2.2} = \frac{10 \pm \sqrt{100 - 104}}{4}$$

$$= \frac{10 \pm \sqrt{-4}}{4} = \frac{10 \pm 2i}{4} = \frac{5 \pm i}{2}$$ which are complex roots.

Illustration 22

Solve the equation : $x^2 - (3\sqrt{2} - 2i)x - 6\sqrt{2}i = 0$

We have $x^2 - (3\sqrt{2} - 2i)x - 6\sqrt{2}i = 0$

Here, a = 1, b = $-(3\sqrt{2} - 2i)$, c = $-6\sqrt{2}i$

Now, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore \quad x = \frac{(3\sqrt{2} - 2i) \pm \sqrt{(3\sqrt{2} - 2i)^2 - 4(1)(-6\sqrt{2}i)}}{2(1)}$

$\therefore$

$$= \frac{3\sqrt{2} - 2i \pm \sqrt{18 - 4 - 12\sqrt{2}i + 24\sqrt{2}i}}{2}$$

$$= \frac{3\sqrt{2} - 2i + \sqrt{(3\sqrt{2} + 2i)^2}}{2} = \frac{3\sqrt{2} - 2i \pm (3\sqrt{2} + 2i)}{2}$$

$$= \frac{6\sqrt{2}}{2}, -\frac{4i}{2} = 3\sqrt{2}, -2i$$

Practice Assignment III

Solve the following equations

1. $2x^2 + 1 = 0$

2. $21x^2 + 9x + 1 = 0$

3. $x^2 + 10ix - 21 = 0$

4. $x^2 - (2\sqrt{3} + 3i)x + 6\sqrt{3}i = 0$

5. $2x^2 + 10 = ix$

6. $x^2 - 7ix - 12 = 0$

7. $i\,x^2 - 4x - 4i = 0$

8. $x^2 - (5 - i)\,x + (18 + i) = 0$

9. $x^2 - (3\sqrt{2} - 2i)\,x - 6\sqrt{2}\,i = 0$

10. $x^2 - (2 + i)\,x - (1 - 7i) = 0$

11. $2x^2 - (3 + 7i)x + (9i - 3) = 0$

12. $x^2 - (7 - i)\,x + (18 - i) = 0$

13. $x^2 - (9 - 2i)\,x + (17 + i) = 0$

14. $2x^2 + ix^2 - 2i = (5 - i)\,x - 2$

15. $abx^2 + (b^2 - ac)ix + bc = 0$

Objective Assignment

1. If a < 0, b > 0, then $\sqrt{a}\,.\sqrt{b}$ is equal to

(a) $-\sqrt{|a|.b}$ **(b)** $\sqrt{|a|.b}\,.i$
(c) $\sqrt{|a|b}$ **(d)** none of these

2. The value of the sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where i = $\sqrt{-1}$, is

(a) i **(b)** i – 1
(c) –i **(d)** 0

3. The smallest positive integral value of n for which $\left(\frac{1-i}{1+i}\right)^n$ is purely imaginary with positive imaginary part, is

(a) 1 **(b)** 3
(c) 5 **(d)** none of these

4. If $(a+ ib)^5 = \alpha + i\beta$, then $(b + ia)^5$ is equal to

(a) $\beta + i\alpha$ **(b)** $\alpha - i\beta$
(c) $\beta - i\alpha$ **(d)** $-\alpha - i\beta$

5. If $z_1 = 9y^2 - 4 - 10ix$, $z_2 = 8y^2 - 20i$, where $z_1 = \bar{z}_2$, then z = x + iy is equal to

(a) – 2 + 2i **(b)** $-2 \pm 2i$
(c) $-2 \pm i$ **(d)** none of these

6. If z is a complex number satisfying the relation |z +1| = z + 2(1 + i), then z is

(a) $\frac{1}{2}(1 + 4i)$ **(b)** $\frac{1}{2}(3 + 4i)$
(c) $\frac{1}{2}(1 - 4i)$ **(d)** $\frac{1}{2}(3 - 4i)$

7. If $e^{i\theta} = \cos\theta + i\sin\theta$, then for the ΔABC, $e^{iA}.\ e^{iB}.\ e^{iC}.$ is

(a) – i **(b)** 1
(c) –1 **(d)** none of these

8. For a complex number z, the minimum value of |z| +|z–2| is

(a) 1 **(b)** 2
(c) 3 **(d)** none of these

9. If $|z_1 - 1| < 1$, $|z_2 - 2| < 2$, $|z_3 - 3| < 3$, then $|z_1 + z_2 + z_3|$

(a) is less than6 **(b)** is more than 3
(c) is less than 12 **(d)** lies between 6 and 12

10. If $z = \frac{\sqrt{3}+i}{\sqrt{3}-i}$, then the fundamental amplitude of z is

(a) $-\frac{\pi}{3}$ **(b)** $\frac{\pi}{3}$
(c) $\frac{\pi}{6}$ **(d)** none of these

11. If z = x + iy satisfies amp(z–1) = amp(z + 3i), then the value of (x – 1) : y is equal to

(a) 2 : 1 **(b)** 1 : 3
(c) –1 : 3 **(d)** none of these

12. $1 + i^2 + i^4 + i^6 + \ldots + i^{2n}$ is :

(a) positive **(b)** negative
(c) 0 **(d)** cannot be determined.

13. If $8iz^3 + 12z^2 - 18z + 27i = 0$, then

(a) $|z| = \frac{3}{2}$ **(b)** $|z| = \frac{2}{3}$
(c) $|z| = 1$ **(d)** $|z| = \frac{3}{4}$

14. If z = 1 + i, then the multiplicative inverse of z^2 is

(a) 1 – i **(b)** $\frac{i}{2}$
(c) $-\frac{i}{2}$ **(d)** 2i

15. The value of $|\sqrt{2i} - \sqrt{-2i}|$ is

(a) 2 **(b)** $\sqrt{2}$
(c) 0 **(d)** $2\sqrt{2}$

16. If z is a complex number such that $|z| \neq 0$ and Re (z) = 0, then:
(a) Re $(z^2) = 0$ **(b)** Im $(z^2) = 0$
(c) $Re(z^2) = Im(z^2)$ **(d)** none of these

17. $\frac{1-ix}{1+ix}$ = a – ib and $a^2 + b^2 = 1$, where a and b are real, then x=

(a) $\frac{2a}{(1+a)^2 + b^2}$ **(b)** $\frac{2b}{(1+a)^2 + b^2}$
(c) $\frac{2a}{(1+b)^2 + a^2}$ **(d)** $\frac{2b}{(1+b)^2 + a^2}$

18. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$, then (x, y)=

(a) (0, 2) **(b)** (–2, 0)
(c) (0, –2) **(d)** none of these

19. If $z_1 = \frac{(\sqrt{3}+i)^2(1-\sqrt{3}i)}{1+i}$ and $z_2 = \frac{(1+\sqrt{3}i)^2(\sqrt{3}-i)}{1-i}$, then

(a) amp z_1 + amp z_2 = 0 **(b)** 3(ampz_1)+ amp z_2= 0
(c) $|z_1| = |z_2|$ **(d)** $3|z_1| = |z_2|$

20. If $z = 1 + i\sqrt{3}$, then $|\arg z| + |\arg \bar{z}|$ equals

(a) $\frac{\pi}{3}$ **(b)** $\frac{2\pi}{3}$
(c) 0 **(d)** $\frac{\pi}{2}$

21. If a, b$\in$ R, then $|e^{a+ib}|$ =
(a) e^a **(b)** e^b
(c) 1 **(d)** none of these

22. The modulus of the complex number $z = \frac{(1-i\sqrt{3})(\cos\theta + i\sin\theta)}{2(1-i)(\cos\theta - i\sin\theta)}$

(a) $\frac{1}{\sqrt{2}}$ **(b)** $\frac{1}{2\sqrt{2}}$

(c) $\frac{1}{\sqrt{3}}$ **(d)** none of these

23. The complex number z satisfying |z–1| = |z–3|=|z–i| is

(a) 2 + i **(b)** $\frac{3}{2}+\frac{1}{2}i$

(c) 2 + 2i **(d)** none of these

24. If $\frac{5z_2}{7z_1}$ is a purely imaginary number, then $\left|\frac{2z_1+3z_2}{2z_1-3z_2}\right|$ is equal to

(a) 5 / 7 **(b)** 7 / 5

(c) 25 / 49 **(d)** none of these

25. The square root of the number – 7 – 24 i is

(a) (3 + 4i) **(b)** (3 – 4i)

(c) ± (3 – 4i) **(d)** none of these

26. Given that the real parts of $\sqrt{5+12i}$ and $\sqrt{5-12i}$ are negative. Then the number

$z = \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$ reduces to

(a) $\frac{3}{2}i$ **(b)** $-\frac{3}{2}i$

(c) $-3+\frac{2}{5}i$ **(d)** none of these

27. $\left(\frac{1+i}{\sqrt{(2)}}\right)^8+\left(\frac{1-i}{\sqrt{(2)}}\right)^8 =$

(a) 1 **(b)** 2

(c) 3 **(d)** 0

28. $\frac{1+7i}{(2-i)^2}$ in (r, θ) form is

(a) ($\sqrt{2}$, π/4) **(b)** ($\sqrt{2}$, π/2)

(c) ($\sqrt{2}$, 3π/4) **(d)** none of these

29. If $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i} = i$, then the real values of x and y are given by

(a) x = –3 y = –1 **(b)** x = 3, y = –1

(c) x = 3, y = 1 **(d)** x = 1, y = –3

30. If (x+iy) (p+iq) = (x^2+y^2) i, then

(a) p = x, q = y **(b)** p = x^2, q = y^2

(c) p = y, q = x **(d)** none of these

Answers

Practice Assignment I

1. (i) $-i$ (ii) 16 (iii) 0 (iv) 2
   (v) $\frac{17-9i}{10}$ (vi) 2i (vii) 1 (viii) 0
   (ix) -1 (x) -1 (xi) 1
2. $-10 - 198i$
3. $-\sqrt{6}$
4. (i) 64 (ii) 1
   (iii) $-2i$ (iv) $\frac{1}{2}+\frac{i}{2}$
   (v) $\frac{1}{8}$
5. (i) 4 + 5 i (ii) $\frac{1}{34}(3+5i)$ (iii) $\frac{1}{2}+\frac{1}{2}i$
   (iv) 2 + 4 i (v) $\frac{3}{5}-\frac{4}{5}i$
6. ………….
7. ………..
8. …………..
9. $2\sqrt{2} + i$
10. x = b, y = a
11. ……
12. 8
13. (i) $-\frac{2}{7}+i\left(\frac{-\sqrt{3}}{7}\right)$ (ii) $\frac{-22}{5}+i\left(\frac{19}{5}\right)$
    (iii) $\frac{63}{25}+i\left(\frac{-16}{25}\right)$ (iv) $-1 + 0i$
    (v) $0+\frac{2b(3a^2-b^2)i}{a^2+b^2}$ (vi) $0 + i\cot\frac{\theta}{2}$
    (vii) $\frac{17}{3}+i\frac{5}{3}$ (viii) -4
    (ix) $\frac{-242}{27}-26i$ (x) $\frac{-22}{3}-i\frac{107}{27}$
14. -160
15. (i) 0, i, $\pm\frac{\sqrt{3}}{2}-\frac{i}{2}$ (ii) 0, $\pm i$.
16. ……
17. ………

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18. (i) $\left(\frac{5}{13}, \frac{14}{13}\right)$ (ii) (3, –1)

(iii) (2, 3), $\left(-2, \frac{1}{3}\right)$ (iv) $\left(\frac{-28}{3}, \frac{-16}{3}\right)$, (–2, 2)

19. ……….
20. (1, –4), (–1, –4)
21. ………..
22. …………..

23. (i) ± (3 + 4i) (ii) ± (2 – i)

(iii) ± (1 + i $\sqrt{3}$) (iv) ± $\frac{1}{\sqrt{2}}$ (1 + i)

(v) ± 2 (1 – i) (vi) ± $\left(\frac{3+4i}{5}\right)$

(vii) ± (4 – 3i) (viii) ± (1 – i)

(ix) ± $\frac{1}{\sqrt{2}}\left[\sqrt{a^2+a+1}+i\sqrt{a^2-a+1}\right]$

(x) ± [(a + b) – i (a – b)]

24. (i) ± 10 (ii) ± 6i

(iii) ± 6

25. ……….

Practice Assignment II

1. (i) $2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ (ii) $\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$

(iii) $\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}$ (iv) $\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$

(v) $\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ (vi) $\sqrt{2}\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right)$

(vii) $2\left(\cos\left(\frac{-2\pi}{3}\right)+i\sin\left(\frac{-2\pi}{3}\right)\right)$ (viii) $2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

(ix) 3(cos π + I sin π) (x) $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

2. (i) 2θ (ii) $\frac{\pi}{3}$

(iii) $\frac{\pi}{6}$ (iv) 0

(v) $\frac{7\pi}{18}$

3. (i) 1 (ii) $\frac{\sqrt{82}}{4}$ (iii) 2

4. (i) 1, $\frac{2\pi}{9}$ (ii) 1, 0

(iii) sec θ, $\theta - \frac{\pi}{2}$ (iv) 2 cos $\left(\frac{\pi}{4}+\frac{\alpha}{2}\right), \left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$

5. 1
6. 0
7. ….
8. 4

PRACTICE ASSIGNMENT III

1. $\pm \frac{i}{\sqrt{2}}$

2. $\frac{-3}{14} \pm \frac{i\sqrt{3}}{42}$

3. –3i, –7i

4. $2\sqrt{3}$, 3i

5. –2i, $\frac{5i}{2}$

6. 3i, 4i

7. –2i, –2i

8. 3 – 4i, 2 + 3i

9. $3\sqrt{2}$, –2i

10. 3 – i, –1 + 2i

11. $\frac{3+i}{2}$, 3i

12. 4 – 3i, 3 + 2i

13. 2 + i, 7 – 3i

14. 1 – i, $\frac{4-2i}{5}$

15. $\frac{ci}{b}, \frac{-bi}{a}$

Objective Assignment

1.	B	11.	B	21.	A
2.	B	12.	D	22.	A
3.	B	13.	A	23.	C
4.	A	14.	C	24.	D
5.	B	15.	A	25.	C
6.	C	16.	B	26.	B
7.	C	17.	B	27.	B
8.	B	18.	C	28.	C
9.	C	19.	B, C	29.	B
10.	B	20.	B	30.	C